Weekly Homework 5

Math 485

October 31, 2013

1. Compute $\phi_X(t) := E\big(\exp(itX)\big)$, where $\exp(x) := e^x$, i is the imaginary number: $i^2 = -1$ and X has $N(\mu, \sigma^2)$ distribution. Recall that the density of $N(\mu, \sigma^2)$ is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

$\phi(t)$ is called the characteristic function of X.

2.
a) Suppose $\mu = 0$. Compute $\phi_X^{(4)}(t)$: the 4th derivative of ϕ_X with respect to t.
b) Use the following fact:

$$E[X^k] = (-i)^k \phi_X^{(k)}(0)$$

to compute $E[(B_t)^4]$ where B is a Brownian motion.

3. Let $0 \le s \le t \le T$ and B a Brownian motion. Compute the followings:
a) $E(B_s^2 B_t)$
b) $E(B_t^2 B_s)$
c) $E(\exp(\sigma B_t - \frac{1}{2}\sigma^2 t))$, where σ is a constant.
d) $E(\exp(\int_0^t \sin(s) dB_s))$.

4. Use Ito formula to compute the following
a) $d\sin(B_t)$
b) $d\exp(B_t)$

5. Let $0 = t_0 < t_1 < t_2 < ... < t_n = T$. Show that

$$E\Big[\big(\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 - T\big)^2\Big] = 2\sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.$$

6. Suppose S_t satisfies

$$dS_t = \sin(S_t)t^2 dt + \exp(\sqrt{S}t - t)dB_t.$$

Compute
a) $d\log(S_t)$
b) $d\exp(S_t^2)$
c) $d\sqrt{S_t}$
Please note that this exercise is only for you to symbolically practice Ito's formula. There may not exist a process S_t that satisfies the above dynamics, or even if it does exist, to make sense in the expression a,b and c.

7. (Extra credit - 5 pts) a) Look up precisely in what "weak sense" the convergence in the definition of Ito integral is. Explain in a few words your understanding of that mode of convergence.
b) Find 3 properties of Brownian motion we have not discussed in the lecture that you find interesting.